Minimum-Redundancy Linear Arrays

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Abstract—**It is shown that there is a class of linear arrays which achieves maximum resolution for a given number of elements by minimizing the number of redundant spacings present in the array. For many-element arrays the degree of redundancy will approach 4/3. Applications of such arrays to aperture synthesis are discussed.**

I. Introduction

THE LINEAR ARRAY is one of the most important types of multi-element antennas, and as such it has played an important role both in communications and in radio astronomy. Existing work on the design of linear arrays has mainly been concerned with the problems of feeding many elements and with tapering the illumination of the aperture to obtain a desired beam shape or degree of sidelobe suppression. However, in arrays designed for observations of faint extragalactic radio sources, the paramount requirement is to maximize the aperture dimensions and the collecting area for a given cost, since the cost will be very high in any event. This paper treats only a portion of the problem by showing that there is a class of arrays which achieve maximum resolution for a given number of elements by reducing the number of redundant spacings present in the array.

Many linear arrays have been used in radio astronomy for observations of the sun. These include the Christiansen grating[1] and the compound interferometer;[2] several examples of each of these types exist at radio observatories around the world. A diagram of a typical grating array is shown in Fig. 1(a). Equal-length branching transmission lines are used to combine the signals from the individual elements at a single receiver input. The spatial-frequency sensitivity diagram for the array is shown in Fig. 1(b). Clearly there is a very high degree of redundancy present. In an N-element grating, unit spacing (equal to u_0 wavelengths) is present $N-1$ times, twice-unit spacing $N-2$ times, and so forth out to the maximum spacing of $N-1$ units, which is present just once. Higher resolution would be achieved if the number of redundant spacings were reduced, permitting the length of the array to be increased.

The high degree of redundancy in the grating array permits the simple feeder arrangement shown in Fig. 1(a), with some modifications if N is not equal to an integral power of two, and produces a directional pattern with desirably low sidelobes. The directional diagram, which has the form $(\sin Nx/N \sin x)^2$, is comblike, with narrow lobes of width[1] $\approx(Nu_0)^{-1}$ which repeat with a grating spacing $\approx u_0^{-1}$. Thus the number of distinct elements which can be resolved in a one-dimensional source distribution is approximately equal to N, and this is obtained when the angular width of the source is equal to the separation between the grating lobes. If the source is an object of known size, such as the sun, the grating spacing should be matched to the source size. For solar observations, adequate signal-to-noise ratios may be achieved with 1- to 5-meter diameter paraboloids as individual elements, and an appropriate unit spacing is $u_0 \gtrsim 100$, since the angular diameter of the sun is about 10^{-2} radian. In any event, the unit spacing determines the size of the field over which the array produces an unambiguous picture of the source distribution.

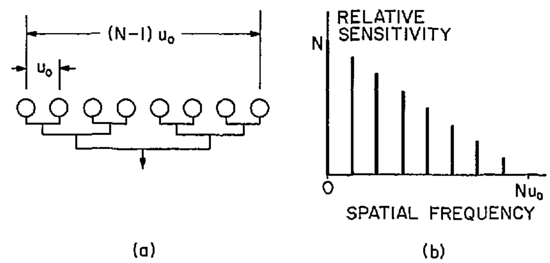

Fig. 1. (a) Grating interferometer with $N=8$. (b) Spatial sensitivity diagram for the grating interferometer.

For observations of faint extragalactic radio sources, antenna arrays must be much larger and more costly. To permit observations of faint sources to be obtained in reasonable lengths of time, collecting areas of the order of 10^4 m^2 are required. In addition, resolutions of the order of seconds of arc are urgently needed, calling for apertures of 10^4 to 10^5 wavelengths in size. Filled apertures of this magnitude are out of the question, but arrays can be built which will achieve the desired sensitivity and resolution. For observations at centimeter and decimeter wavelengths, the optimum choice[3] for the element of such an array is a steerable paraboloid of 15 to 45 m. The considerations which lead to the choice of a particular element *size* are beyond the scope of this paper, the purpose of which is to show that for a given *number* of elements there is an optimum linear array giving maximum resolution. This is obtained by minimizing the number of redundant spacings present in the array.

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[1] The actual width of the lobes in this array is closer to $2(Nu_0)^{-1}$, since the low spatial-frequency components are so heavily weighted. This difference is neglected in the subsequent comparisons.

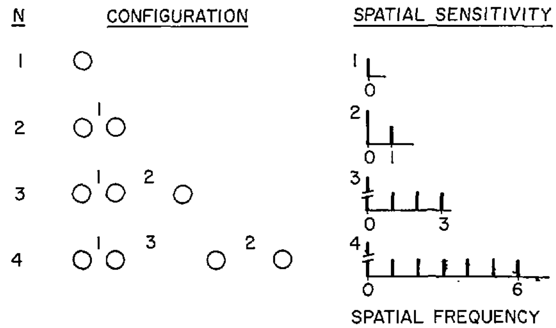

Fig. 2. The four zero-redundancy linear arrays and their spatial sensitivity diagrams. Except for the zero spacing, each spacing is present once and only once.

II. Minimum-Redundancy Arrays

The minimum-redundancy array was considered by Arsac,[4] who constructed the largest possible linear array with zero redundancy. There are four such arrays, as shown in Fig. 2. The first is the trivial case of a single-element "array." In the others there is one, and only one, pair of elements separated by each multiple of the unit spacing out to a maximum spacing equal to the distance between the end elements. Thus each of these arrays uniformly samples the spatial-frequency spectrum out to a spacing $u_{max}=\frac{1}{2}N(N-1)u_0=N_{max}u_0$, where $\frac{1}{2}N(N-1)$ is the number of possible pairs of N distinct elements. Bracewell[5] has given an elegant proof that these are the only linear arrays having zero redundancy.

It can be seen from Fig. 2 that the zero-redundancy arrays sample the spatial-frequency spectrum at uniform intervals and with uniform sensitivity except for the zero spacing, or total power, component. The reception pattern which results has approximately the form $[(\sin 2Nx/2N \sin x)+constant]$ and gives the highest possible resolution for a given aperture length. The result of scanning a source distribution with such a reception pattern is to reject all spatial components in the source with spatial frequency greater than the maximum to which the array is sensitive, yielding an image of the source known as the *principal solution.*[6] It is not necessary that the array sample the spatial-frequency spectrum of the source at exactly uniform intervals, but the grating sidelobes will become serious at a distance from the main lobe equal to the inverse of the *maximum* spacing between samples (measured in wavelengths). Processing of the data from an array is very much simplified if the sampling is done at uniform intervals, and the arrays considered below retain this property.

In an array with more than four elements, it is clear that there must be some configuration of the elements which leads to minimum redundancy while still retaining uniform coverage of the spatial-frequency spectrum. A firm upper limit for the minimum redundancy is set by the rather simple division of the elements of the array into two equal groups, shown schematically in Fig. 3(a) for the case $N=8$. This is the compound grating interferometer; a good example of this type of radio telescope has been described by Picken and Swarup.[6] With the branching feeder arrangement shown in Fig. 3(a), the receiver sees the correlated signal between the narrow-spaced half of the elements and the wide-spaced half, giving uniform coverage of the spatial-frequency spectrum out to $\frac{1}{4}N^2u_0$ (for even values of N). By accepting a more complex receiving arrangement in which the signals from the antennas are individually correlated with each other, the spacing between elements in the wide-spaced half may be increased to $(\frac{1}{2}N+1)u_0$, giving coverage out to $(\frac{1}{4}N^2+\frac{1}{2}N-1)u_0$. This arrangement is shown in Fig. 3(b).

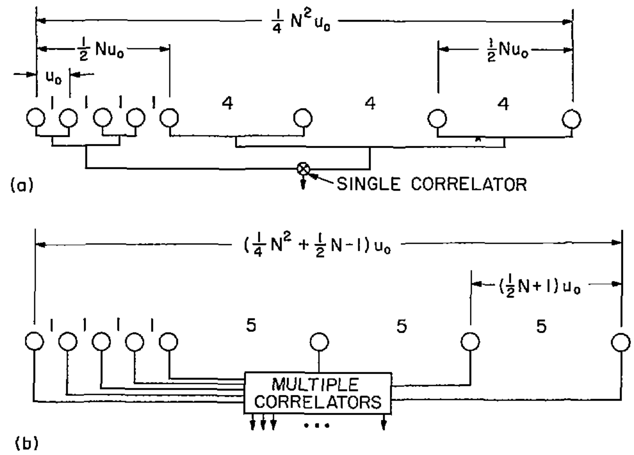

Fig. 3. (a) Compound grating interferometer with branching feeds and a single correlation detector. (b) Modified compound grating interferometer in which higher resolving power is obtained by eliminating some of the redundant spacings present in the simple compound grating shown in (a). A separate correlator is used for each pair of antennas.

The advantage of the arrangement in Fig. 3(b) is appreciable only for modest values of N; for large N, the aperture width tends toward $\frac{1}{4}N^2u_0$ for either arrangement. Since the number of distinct pairs $\frac{1}{2}N(N-1)$ tends toward $\frac{1}{2}N^2$, the redundancy factor in these compound grating arrays is always ≤ 2. Numerically, the redundancy R may be defined by the ratio of the number of pairs to N_{max},

$$R=\frac{\frac{1}{2}N(N-1)}{N_{max}},$$

where N_{max} is the greatest multiple of the unit spacing such that all multiples of the unit spacing $\leq N_{max}$ are present between pairs of elements in the array. For large N, the redundancy thus approaches $N^2/(2N_{max})$. The redundancies for the particular eight-element compound gratings shown in Fig. 3(a) and (b) are 1.75 and 1.47, respectively.

The problem of finding the array configuration giving the lowest possible redundancy for a given value of N is not a simple one. It happens that this identical problem is of some interest in the theory of numbers, and it has been examined by Leech,[7] who gives some solutions for $N\leq 11$ and demonstrates that in the limit of large N, the minimum redundancy will lie between 1.217 and 1.332. The details of Leech's proof are not required here; for convenience we may take 4/3 as

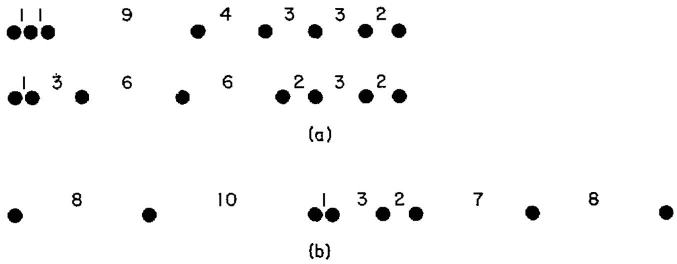

Fig. 4. (a) The two restricted minimum-redundancy eight-element arrays, which uniformly cover the range of 1 to 23 times the unit spacing. (b) The general minimum-redundancy eight-element array, which uniformly covers 1 to 24 times the unit spacing. This array has a length of 39 times the unit spacing. The extra spacings are 8, 31 (twice), and 39 times the unit spacing.

an estimate of the minimum redundancy in an array with a very large number of elements, say twenty or more.

At this point we may distinguish between two possible cases. In the *restricted* case the spatial-frequency spectrum is uniformly covered up to a spacing $N_{\max}u_0$, and this is also the distance between the end elements of the array. The redundant spacings are all less than the maximum spacing. In the *general* case the length of the array may be greater than $N_{\max}u_0$. The remaining spacings, which number $\frac{1}{2}N(N-1)-N_{\max}$, are then not all redundant, since some exceed $N_{\max}u_0$, but the coverage of the spatial-frequency spectrum is uniform only out to $N_{\max}u_0$. As an example, Fig. 4 shows the two optimum restricted eight-element arrays and the optimum general eight-element array. In the former, which have configurations ·1·1·9·4·3·3·2· and ·1·3·6·6·2·3·2·, the value of $N_{\max}$ is 23, and the length of the array is $23u_0$. In the latter, which has the configuration ·8·10·1·3·2·7·8·, the value of $N_{\max}$ is 24, and the length is $39u_0$. The redundancies for these eight-element arrays are 1.22 and 1.17, respectively.

In some applications, the additional resolution afforded by the larger overall length of the general array might be welcome, even though the spatial-frequency coverage is incomplete above $N_{\max}u_0$. In other applications, there may not be sufficient space to provide the extra array length, and the restricted array will be more attractive.

Comparing the eight-element arrays of Figs. 1 and 4, we see that the latter gives more than three times greater coverage of the spatial-frequency spectrum for equal values of the unit spacing u_0, i.e., for equal grating lobe separations. The cost of this greater coverage is a more complicated signal processing system and higher sidelobes. However, the signal processing system will almost certainly include manipulations of the correlator outputs in a digital computer, at which point the beam may be tailored to have any desired sidelobe level by choosing appropriate weighting of the various correlator outputs representing different element separations. An additional advantage of this system is that in the computer the electrical path lengths from each antenna to the correlators may be determined after the fact and corrections may be applied for any changes in these lengths. This eliminates the tedious adjustment required in a grating array

TABLE I

Some Minimum-Redundancy Array Configurations

N	$N_{\max}$	R	Configuration
Restricted Arrays:			
5	9	1.11	·1·3·3·2·
6	13	1.16	·1·5·3·2·2
7	17	1.24	·1·3·6·2·3·2·
8	23	1.22	·1·3·6·6·2·3·2·
9	29	1.24	·1·3·6·6·6·2·3·2·
10	36	1.25	·1·2·3·7·7·7·4·4·1·
11	43	1.30	·1·2·3·7·7·7·7·4·4·1·
General Arrays:			
5	9	1.11	·4·1·2·6·
6	13	1.16	·6·1·2·2·8·
7	18	1.17	·14·1·3·6·2·5·
8	24	1.17	·8·10·1·3·2·7·8·
10	37	1.22	·16·1·11·8·6·4·3·2·22·
11	45	1.24	·18·1·3·9·11·6·8·2·5·28·

in which the signals from each antenna must be added together in exactly the right phase before detection.[8]

A disadvantage of the minimum-redundancy array is that its resolution is not easily increased except by adding more elements and rearranging the array to the optimum configuration for the new number of elements. With the compound grating arrays, increased resolution is readily obtained by combining observations with different array configurations. For instance, the resolution of the array in Fig. 3(a) may be doubled by taking an additional observation with the separation between the narrow-spaced and wide-spaced halves of the array increased by an amount equal to the length of the original array, $\frac{1}{4}N^2u_0$. The two observations are combined coherently to achieve double the original resolution with the same grating lobe spacing.

It is not an easy matter to work out the minimum-redundancy configuration for a large value of N. Several short cuts are described by Leech. Those who might be tempted to try a "brute force" solution with a digital computer should recall that the number of possible configurations even in the restricted case is approximately $\frac{1}{2}(N_{\max}-2)!/[(N-2)!(N_{\max}-N)!]$ or about 6×10^{10} for the optimum ten-element array with $N_{\max}=36$. Table I gives examples of restricted and general array configurations for $N=5$ to 11, selected from the more extensive table compiled by Leech.[7] The patterns which seem to be apparent in the restricted arrays for $N=7$ to 11 do not repeat. There is no nine-element general array with greater length than the optimum restricted array of nine elements.

III. Applications to Aperture Synthesis

In radio astronomy, resolution in more than a single dimension is usually desired, and two-dimensional arrays[9]–[11] or aperture synthesis schemes[12] have often been used to achieve two-dimensional resolution with combinations of linear arrays. The minimum-redundancy arrays discussed here are not directly applicable to two-dimensional antennas. Indeed, Bracewell[6] has already noted that there is no two-dimensional analog to the zero-redundancy linear arrays of Fig. 1 except for the trivial case of a four-element tee.

However, two-dimensional resolution may be obtained in radio astronomical observations over a substantial part of the sky using a single linear array and the *earth rotation synthesis* technique pioneered by Ryle.[13],[14] In this technique, a source is observed over a period of twelve hours, during which time the rotation of the earth changes the orientation of the linear array with respect to the source, giving the coverage in the spatial-frequency plane necessary to reconstruct a two-dimensional picture of the source. The method works best for a source located at the celestial pole, where the reconstructed beam is circular. For sources with smaller declinations δ the resolution in the north-south direction is degraded by a factor $\csc \delta$ if the baseline of the array runs east and west. It is, of course, presumed that the properties of the source do not change during the period of observation.

Minimum-redundancy linear arrays may readily be used with the earth rotation synthesis technique to obtain two-dimensional resolution. The resultant pencil beam has an angular diameter $\approx (N_{max} u_0)^{-1}$, and the first grating sidelobe is a ring with radius u_0^{-1}. Thus the array of Fig. 4(b) with a unit spacing of $u_0 = 10^3$ would produce a picture of an area 6.8 minutes of arc in diameter with a resolution of the order of 10 seconds of arc. It would achieve this with 8 antennas and 24 correlators (or a maximum of 28 if all possible combinations were recorded) in an observing time of twelve hours. For comparison, a conventional grating tee with comparable field size and resolution would require $3 \times 22 + 1 = 67$ antennas and a minimum of $22 \times 47 = 1034$ correlators if separate products were taken to avoid the phase-adjustment problem inherent in signal combination prior to detection. The observation with the tee, however, would take only as much time as was required to attain the desired signal-to-noise ratio.

It is clear that the tee would be very much more complicated and costly than a synthesis array of equivalent resolution. However, the rate at which information is collected by the tee would be considerably higher, particularly on intense objects for which the tee would not require long integration times. For faint point sources, the rate of information collection would be proportional to the total collecting area in each array, although the tee would require a correlator between each possible pair of elements (a total of 2211) in order to fully utilize its collecting area.

A variable spacing eight-element interferometer, which could assume any of the linear configurations discussed in this paper, as well as tee and cross configurations, has been proposed for construction at the Owens Valley Radio Observatory.[15] The National Science Foundation has sponsored the construction of the first element of this array, a 40 m (130 ft) diameter paraboloid now nearing completion. If the remainder of the array is approved for construction, it will become an extremely powerful instrument for the study of faint radio sources.

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